Asymptotic controllability of piecewise linear systems

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(asymptotic) controllability problem

$$\dot{x} = f(x, u) \qquad \begin{cases} x : \text{ state} \\ u : \text{ input} \end{cases}$$

controllability

$$\begin{tikzcd} x_0 \arrow[r, dashed, "u"] \arrow[d, equal] & x_f \arrow[d, equal] \\ x(0) & x(T) \end{tikzcd}$$

asymptotic controllability

$$\begin{tikzcd} x_0 \arrow[r, dashed, "u"] \arrow[d, equal] & 0 \arrow[d, equal] \\ x(0) & x(\infty) \end{tikzcd}$$

linear vs. nonlinear

linear	piecewise linear	nonlinear
well-understood	?	typically local results

piecewise linear systems: despair

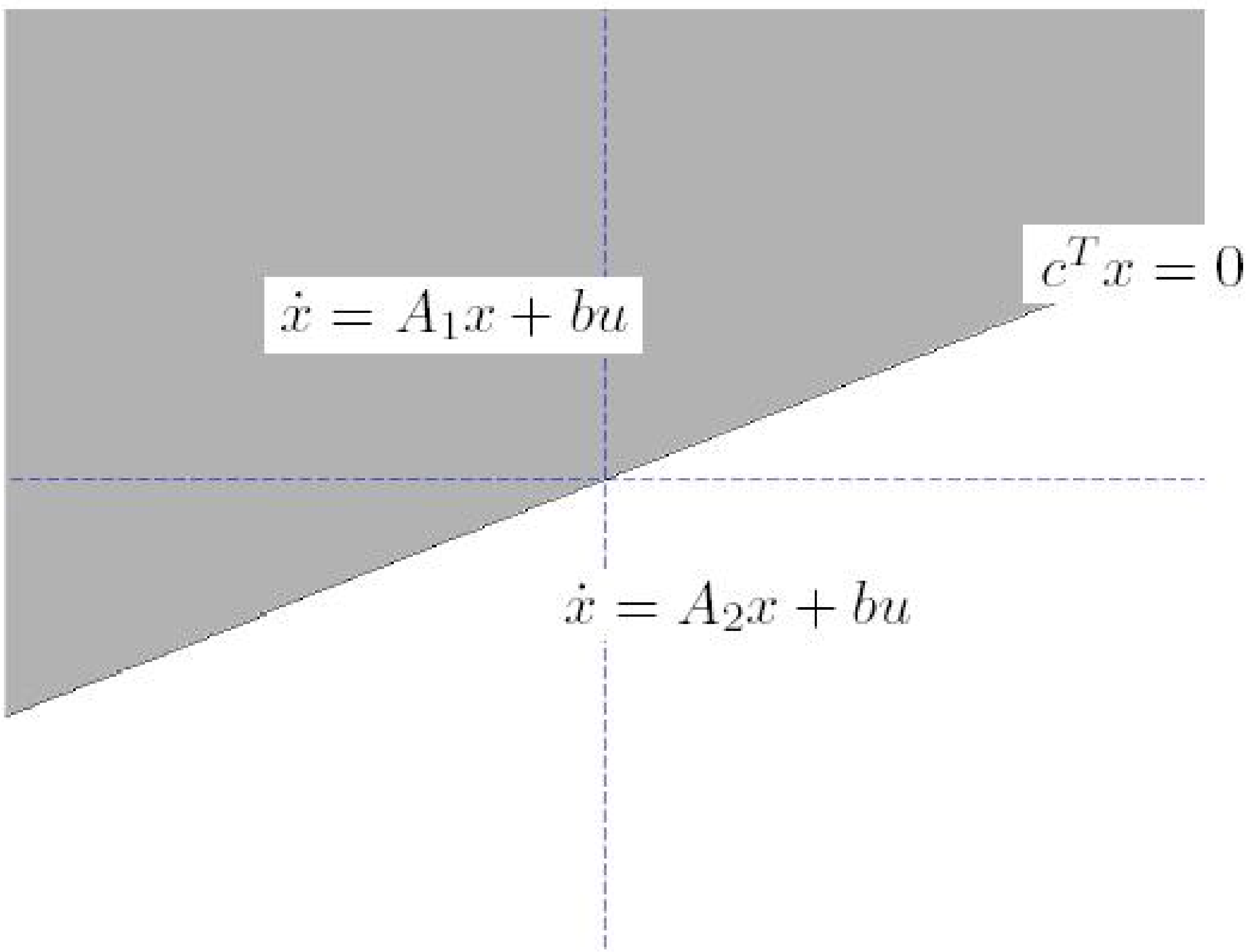
Thm. [Blondel and Tsitsiklis]

The *null-controllability* and *reachability* of the discrete-time sign system

$$x(k+1) = \begin{cases} A_{-}x(k) + bu(k) & \text{if } c^T x(k) < 0 \\ A_0 x(k) + bu(k) & \text{if } c^T x(k) = 0 \\ A_{+}x(k) + bu(k) & \text{if } c^T x(k) > 0 \end{cases}$$

are **UNDECIDABLE!**

bimodal piecewise linear systems



bimodal piecewise linear systems

$$\dot{x} = \begin{cases} A_1 x + bu & c^T x \geqslant 0 \\ A_2 x + bu & c^T x \leqslant 0 \end{cases} \qquad \begin{matrix} x \in \mathbb{R}^n \\ u \in \mathbb{R} \end{matrix}$$

Continuity:

$$\boxed{c^T x = 0 \Longrightarrow A_1 x = A_2 x}$$

$$\Updownarrow$$

$$\boxed{A_1 - A_2 = ec^T \text{ for some } e}$$

controllability of linear systems

Thm. [Kalman]

The linear system

$$\dot{x} = Ax + Bu$$

1. is controllable if, and only if, the following implication holds

$$\lambda \in \mathbb{C}, \quad 0 \neq v \in \mathbb{C}, \quad v^* A = \lambda v^* \quad \Longrightarrow \quad v^* B \neq 0.$$

2. is asymptotically controllable if, and only if, the following implication holds

$$\lambda \in \mathbb{C}_+, \quad 0 \neq v \in \mathbb{C}, \quad v^* A = \lambda v^* \quad \Longrightarrow \quad v^* B \neq 0.$$

positive controllability of linear systems

Thm. [Brammer]

The linear system

$$\dot{x} = Ax + Bu$$

with $u(t) \geqslant 0$ for all t is controllable if, and only if,

1. (A, B) is controllable, and

2. the implication

$$\lambda \in \mathbb{R}, \quad 0 \neq v \in \mathbb{R}, \quad v^T A = \lambda v^T \quad \Longrightarrow \quad v^T B \not\geqslant 0.$$

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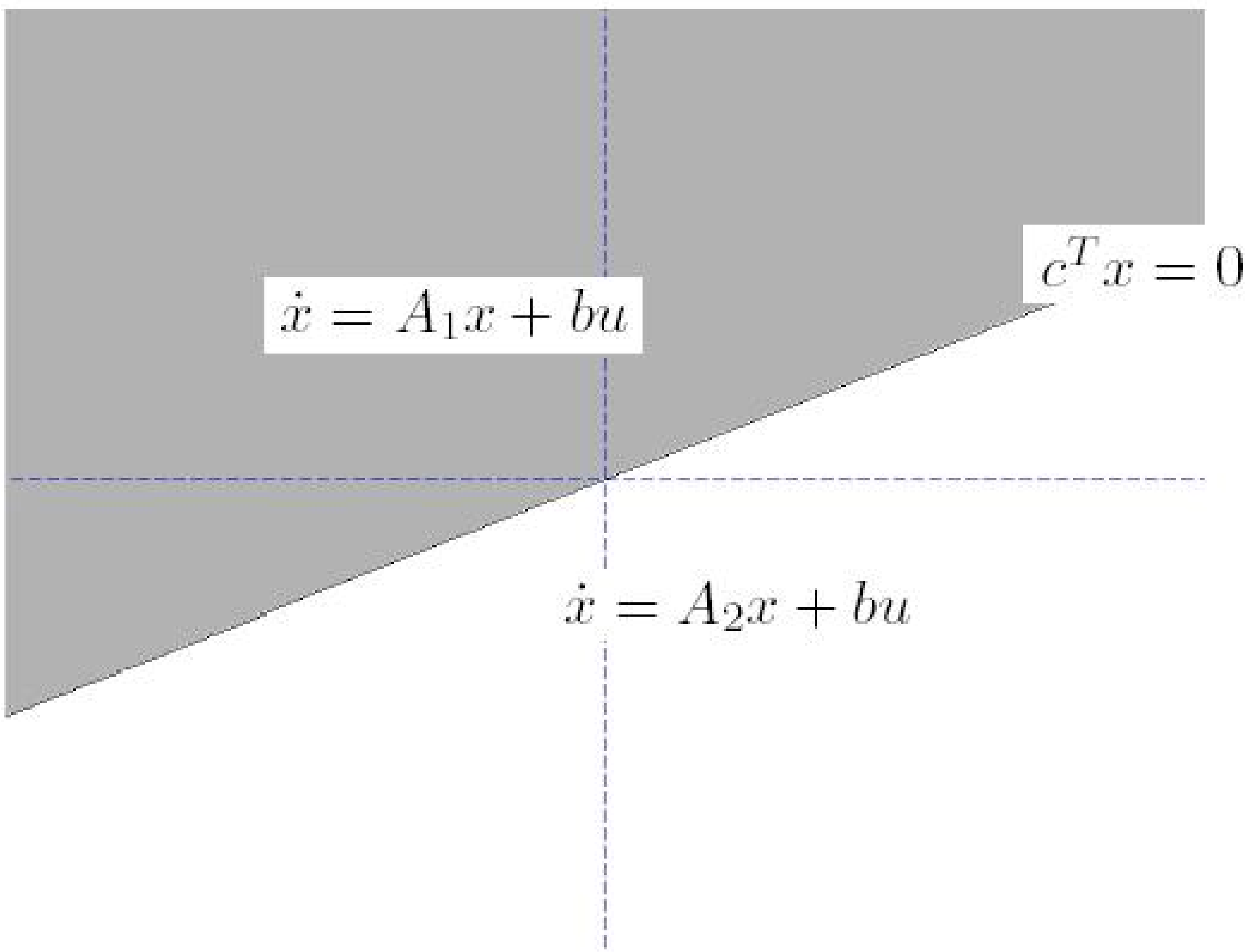
single input case: A must not have any real eigenvalues!

bimodal piecewise linear systems



an example

Consider the bimodal system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 + u \end{aligned} \qquad \text{if } y_1 = x_2 \geqslant 0$$

$$\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 + u \end{aligned} \qquad \text{if } y_2 = x_2 \leqslant 0$$

Both modes are controllable but NOT the overall system!

another example

Consider the bimodal system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -x_3 + x_5 \\ \dot{x}_5 &= u \end{aligned} \qquad \text{if } y_1 = x_5 \geqslant 0$$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + x_5 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -x_3 \\ \dot{x}_5 &= u \end{aligned} \qquad \text{if } y_2 = x_5 \leqslant 0$$

Both modes are UNcontrollable but the overall system is controllable!

a sufficient condition

Consider the bimodal system

$$\begin{aligned}\dot{x}_1 &= x_2 + u \\ \dot{x}_2 &= x_1\end{aligned} \quad \text{if } x_2 \geqslant 0 \qquad \frac{X_2(s)}{U(s)} = \frac{1}{s^2 - 1}$$

$$\begin{aligned}\dot{x}_1 &= -x_2 + u \\ \dot{x}_2 &= x_1\end{aligned} \quad \text{if } x_2 \leqslant 0 \qquad \frac{X_2(s)}{U(s)} = \frac{1}{s^2 + 1}$$

$$u = \begin{cases} (\frac{d^2}{dt^2} - 1)x_2 & \text{if } x_2 \geqslant 0 \\ (\frac{d^2}{dt^2} + 1)x_2 & \text{if } x_2 \leqslant 0 \end{cases} \qquad \begin{array}{ll} x_2(0) = x_{20} & x_2(1) = x_{2f} \\ \dot{x}_2(0) = x_{10} & \dot{x}_2(1) = x_{1f}. \end{array}$$

Polynomial inverse ⇒ controllability!

invariant zeros

Consider the linear system

$$\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx + Du.\end{aligned}$$

A complex number $\lambda \in \mathbb{C}$ is an *invariant zero* if

$$\mathbf{rank}_{\mathbb{R}} \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} < \mathbf{rank}_{\mathbb{R}[s]} \underbrace{\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}}_{\text{system matrix}}$$

an implication of continuity

$$A_1 - A_2 = ec^T.$$

$$\begin{bmatrix} A_1 - \lambda I & b \\ c^T & 0 \end{bmatrix} = \begin{bmatrix} I & e \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A_2 - \lambda I & b \\ c^T & 0 \end{bmatrix}$$

invariant zero directions

A vector $\mathrm{col}(v, \mu) \in \mathbb{C}^{n+m}$ is said to be a *(left) zero direction* for the invariant zero $\lambda \in \mathbb{C}$ if

$$\begin{bmatrix} v^* & \mu^* \end{bmatrix} \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} = 0.$$

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right zero directions: if

$$\begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} \begin{bmatrix} x_0 \\ \bar{u} \end{bmatrix} = 0$$

then $y = Cx + Du \equiv 0$ for $x(0) = x_0$ and $u(t) = \exp(\lambda t)\bar{u}$.

a necessary condition

$$\begin{bmatrix} v^T & \mu_i \end{bmatrix} \begin{bmatrix} A_i - \lambda I & b \\ c^T & 0 \end{bmatrix} = 0 \qquad (\lambda \in \mathbb{R})$$

$$v^T A_i = \lambda v^T - \mu_i c^T$$

$$v^T \dot{x} = v^T \begin{cases} A_1 x + bu & \text{if } c^T x \geqslant 0 \\ A_2 x + bu & \text{if } c^T x \leqslant 0 \end{cases}$$

$$= \lambda v^T x - \begin{cases} \mu_1 c^T x & \text{if } c^T x \geqslant 0 \\ \mu_2 c^T x & \text{if } c^T x \leqslant 0 \end{cases}$$

$\mu_1 \mu_2 > 0$ is a necessary condition for controllability!

controllability of bimodal systems

Thm.

The bimodal system

$$\dot{x} = \begin{cases} A_1 x + bu & \text{if } c^T x \geqslant 0, \\ A_2 x + bu & \text{if } c^T x \leqslant 0 \end{cases}$$

where $A_1 - A_2 = ec^T$ is controllable if, and only if,

1. $(A_1, \begin{bmatrix} b & e \end{bmatrix})$ is controllable $(\Leftrightarrow \langle A_1 \mid \operatorname{im} b \rangle + \langle A_2 \mid \operatorname{im} b \rangle = \mathbb{R}^n)$ and
2. the implication

$$\begin{bmatrix} v^T & \mu_i \end{bmatrix} \begin{bmatrix} A_i - \lambda I & b \\ c^T & 0 \end{bmatrix} = 0, i \in \{1, 2\}, \lambda \in \mathbb{R} \Rightarrow \mu_1 \mu_2 > 0$$

holds.

flavor of the proof

$\mathcal{V}_i^*(A_i, b, c^T, 0)$	$\mathcal{T}_i^*(A_i, b, c^T, 0)$
the largest of $\mathcal{V}_i$s such that	the smallest of $\mathcal{T}_i$s such that
$(A_i - bf^T)\mathcal{V}_i \subseteq \mathcal{V}_i$	$(A_i - gc^T)\mathcal{T}_i \subseteq \mathcal{T}_i$
$\mathcal{V}_i \subseteq \ker c^T$	$\operatorname{im} b \subseteq \mathcal{T}_i$
for some $f \in \mathbb{R}^n$	for some $g \in \mathbb{R}^n$

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continuity: $\mathcal{V}^* := \mathcal{V}_1^* = \mathcal{V}_2^*$ and $\mathcal{T}^* := \mathcal{T}_1^* = \mathcal{T}_2^*$!

push-pull systems

Geometric control theory: $c^T(sI - A)^{-1}b \not\equiv 0 \Rightarrow \mathbb{R}^n = \mathcal{V}^* \oplus \mathcal{T}^*$.

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after a coordinate change and input transformation:

$$\begin{aligned}
\dot{x}_1 &= Kx_1 + \begin{cases} g_1 c_2^T x_2 & \text{if } c_2^T x_2 \geqslant 0 \\ g_2 c_2^T x_2 & \text{if } c_2^T x_2 \leqslant 0 \end{cases} \\
\dot{x}_2 &= b_2 u' + \begin{cases} L_1 x_2 & \text{if } c_2^T x_2 \geqslant 0 \\ L_2 x_2 & \text{if } c_2^T x_2 \leqslant 0 \end{cases} \\
y &= c_2^T x_2
\end{aligned}$$

$\mathcal{V}^*(L_i, b_2, c_2^T, 0) = \{0\} \Rightarrow c_2^T(sI - L_i)^{-1}b_2$ has a polynomial inverse.

controllability of push-pull system

Thm. The following statements are equivalent.

1. The system $\dot{\zeta} = M\zeta + \begin{cases} N_1\eta & \text{if } \eta \geqslant 0 \\ N_2\eta & \text{if } \eta \leqslant 0 \end{cases}$ is controllable.

2. There exists no nonzero vector ξ such that $\xi^T e^{Mt} N_1 \leqslant 0$ and $\xi^T e^{Mt} N_2 \geqslant 0$ for all $t \geqslant 0$.

3. The following conditions hold.
   (a) $(M, \begin{bmatrix} N_1 & N_2 \end{bmatrix})$ is controllable and
   (b) the implication

$$v^T M = \lambda v^T,\ \lambda \in \mathbb{R},\ v \neq 0 \Rightarrow (v^T N_1)(v^T N_2) > 0$$

holds.

special cases (i)

$$\dot{\zeta} = M\zeta + \begin{cases} N_1\eta & \text{if } \eta \geqslant 0 \\ N_2\eta & \text{if } \eta \leqslant 0 \end{cases}$$

$$N_1 = N_2 = N$$

$$\dot{x} = Mx + Nu$$

$$u \in \mathbb{R}$$

controllable

$$\Updownarrow$$

(M, N) is controllable

special cases (ii)

$$\dot{\zeta} = M\zeta + \begin{cases} N_1\eta & \text{if } \eta \geqslant 0 \\ N_2\eta & \text{if } \eta \leqslant 0 \end{cases}$$

$N_1 = -N_2 = N$ (alternatively, $N_1 = N$ and $N_2 = 0$)

$$\dot{x} = Mx + Nu$$

$$u \in \mathbb{R}_+$$

controllable

$$\Updownarrow$$

1) (M, N) is controllable

2) M has no real eigenvalues.

bimodal systems (revisited)

Thm.

The bimodal system

$$\dot{x} = \begin{cases} A_1 x + bu & \text{if } c^T x \geqslant 0, \\ A_2 x + bu & \text{if } c^T x \leqslant 0 \end{cases}$$

where $A_1 - A_2 = ec^T$ is controllable if, and only if,

1. $(A_1, \begin{bmatrix} b & e \end{bmatrix})$ is controllable $(\Leftrightarrow \langle A_1 \mid \mathrm{im}\, b \rangle + \langle A_2 \mid \mathrm{im}\, b \rangle = \mathbb{R}^n)$ and
2. there exist $\lambda \in \mathbb{R}$, $v \in \mathbb{R}^n$, and $\mu_i \in \mathbb{R}$ such that

$$\begin{bmatrix} v^T & \mu_i \end{bmatrix} \begin{bmatrix} A_i - \lambda I & b \\ c^T & 0 \end{bmatrix} = 0, i = 1, 2 \Rightarrow \mu_1 \mu_2 > 0.$$

asymptotic controllability

Thm.

The bimodal system

$$\dot{x} = \begin{cases} A_1 x + bu & \text{if } c^T x \geqslant 0, \\ A_2 x + bu & \text{if } c^T x \leqslant 0 \end{cases}$$

where $A_1 - A_2 = ec^T$ is asymptotically controllable if, and only if,

1. $(A_1, \begin{bmatrix} b & e \end{bmatrix})$ is asymptotically controllable and
2. there exist $\lambda \in \mathbb{R}_+$, $v \in \mathbb{R}^n$, and $\mu_i \in \mathbb{R}$ such that

$$\begin{bmatrix} v^T & \mu_i \end{bmatrix} \begin{bmatrix} A_i - \lambda I & b \\ c^T & 0 \end{bmatrix} = 0, i = 1, 2 \Rightarrow \mu_1 \mu_2 > 0.$$

conewise linear systems

$$\begin{aligned}\dot{x} &= Ax + Bu + f(y) \\ y &= Cx + Du \in \mathcal{Y}\end{aligned} \qquad \begin{aligned} x &\in \mathbb{R}^n \\ u &\in \mathbb{R}^m \\ y &\in \mathbb{R}^p\end{aligned}$$

where

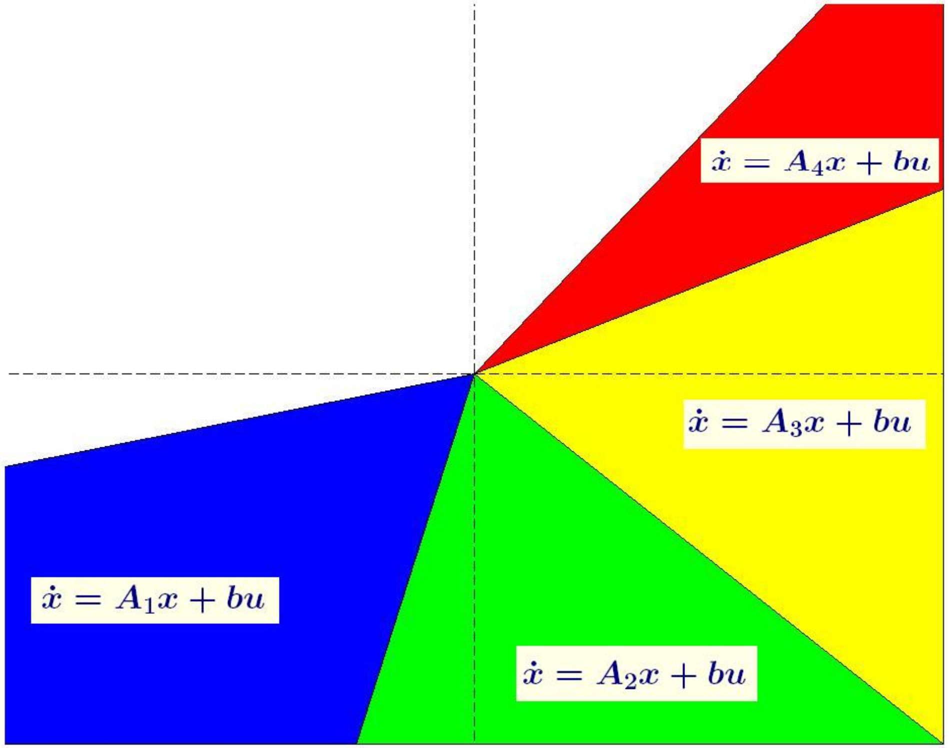
- $\mathcal{Y} = \cup_{i=1}^{r} \mathcal{Y}_i$,
- f is a *conewise linear function on* $\mathcal{Y}$,

$$f(y) = M^i y \text{ if } y \in \mathcal{Y}_i.$$

alternatively

$$\dot{x} = (A + M^i C)x + (B + M^i D)u \quad \text{if } Cx + Du \in \mathcal{Y}_i$$

$$Cx + Du \in \mathcal{Y} = \cup_{i=1}^{r} \mathcal{Y}_i$$



examples of conewise linear systems

- bimodal systems
- complementarity systems

$$
\begin{aligned}
\dot{x} &= Ax + Bu + Ez \\
w &= Cx + Du + Fz \\
\mathcal{C} \ni z &\perp w \in \mathcal{C}^*
\end{aligned}
$$

  - mechanics: contact and friction
  - electrical eng.: circuits with switching and/or piecewise elements (power converters)
  - control theory: optimal control problems with unilateral state constraints
  - math. programming: dynamic optimization subject to inequality constraints
  - math. finance: pricing of derivatives with early exercise opportunities

controllability of conewise linear systems

Thm.

Consider the conewise linear system

$$\dot{x} = (A + M^i C)x + (B + M^i D)u \quad \text{if } Cx + Du \in \mathcal{Y}_i \tag{1a}$$

$$Cx + Du \in \mathcal{Y} \tag{1b}$$

$$\mathcal{Y} = \cup_{i=1}^{r} \mathcal{Y}_i \tag{1c}$$

Suppose that $D + C(sI - A)^{-1}B$ is invertible. The CLS (1) is controllable if, and only if, the following implications hold:

1. there exist $\lambda \in \mathbb{C}$, $v \in \mathbb{C}^n$, and $\mu_i \in \mathbb{C}^m$ such that

$$\begin{bmatrix} v^* & \mu_i^* \end{bmatrix} \begin{bmatrix} A + M^i C - \lambda I & B + M^i D \\ C & D \end{bmatrix} = 0 \text{ and } \mu_i = 0 \text{ for all } i \Rightarrow z = 0$$

2. there exist $\lambda \in \mathbb{R}$, $v \in \mathbb{R}^n$, and $\mu_i \in \mathbb{R}^m$ such that

$$\begin{bmatrix} v^T & \mu_i^T \end{bmatrix} \begin{bmatrix} A + M^i C - \lambda I & B + M^i D \\ C & D \end{bmatrix} = 0 \text{ and } \mu_i \in \mathcal{Y}_i^* \text{ for all } i \Rightarrow z = 0$$

asy. cont. of conewise linear systems

Thm.

Consider the conewise linear system

$$\dot{x} = (A + M^i C)x + (B + M^i D)u \quad \text{if } Cx + Du \in \mathcal{Y}_i \tag{2a}$$

$$Cx + Du \in \mathcal{Y} \tag{2b}$$

$$\mathcal{Y} = \cup_{i=1}^{r} \mathcal{Y}_i \tag{2c}$$

Sps. that $D + C(sI - A)^{-1}B$ is invertible. The CLS (2) is asymptotically controllable if, and only if, the following implications hold:

1. there exist $\lambda \in \mathbb{C}_+$, $z \in \mathbb{C}^n$, and $w_i \in \mathbb{C}^m$ such that

$$[z^* w_i^*] \begin{bmatrix} A + M^i C - \lambda I & B + M^i D \\ C & D \end{bmatrix} = 0 \text{ and } w_i = 0 \text{ for all } i \Rightarrow z = 0$$

2. there exist $\lambda \in \mathbb{R}_+$, $z \in \mathbb{R}^n$, and $w_i \in \mathbb{R}^m$ such that

$$[z^T w_i^T] \begin{bmatrix} A + M^i C - \lambda I & B + M^i D \\ C & D \end{bmatrix} = 0 \text{ and } w_i \in \mathcal{Y}_i^* \text{ for all } i \Rightarrow z = 0$$

conclusions and future work

- combination of ideas from geometric control theory and mathematical programming

- continuity!

- more general classes of piecewise linear systems

- feedback stabilization of piecewise linear systems
  (perhaps by continuous feedback!)
  - linear systems:
    controllability $\Rightarrow$ asymptotic controllability $\Rightarrow$ feedback stabilizability
  - nonlinear systems:
    asymptotic controllability $\not\Rightarrow$ continuous feedback stabilizability
    asymptotic controllability $\Rightarrow$ discontinuous feedback stabilizability