Higher order Sweeping process: Mathematical and numerical aspects

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Let us here consider the following dynamical system:

$$\begin{cases} \dot{x}(t) = Ax(t) + B\lambda(t) \quad (t \geq 0) \\ \\ x(0) = x_0 \\ \\ w(t) = Cx(t) \geq 0 \quad (t \geq 0) \end{cases} \tag{1}$$

where $x : \mathbb{R}^+ \to \mathbb{R}^n$, $\lambda \in \mathbb{R}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$ and $C \in \mathbb{R}^{1\times n}$.

Objective: embedd (1) in a suitable formalism to integrate and discretize with a time-stepping scheme.

Original motivation:

Let us consider the following case:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} ; B = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} ; C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} ; D = 0 \tag{2}$$

and let us embedd the system into Linear Complementarity Systems

$$\begin{cases} \dot{x}(t) = Ax(t) + B\lambda(t) \\ x(0) = x_0 \\ 0 \leq w(t) = Cx(t) \perp \lambda(t) \geq 0 \end{cases} \tag{3}$$

Then applying the backward Euler scheme:

$$\begin{cases} \dfrac{x_{k+1} - x_k}{h} = Ax_{k+1} + B\lambda_{k+1} \\ w_{k+1} = Cx_{k+1} + D\lambda_{k+1} \\ 0 \leq \lambda_{k+1} \perp w_{k+1} \geq 0 \end{cases} \tag{4}$$

we obtain with $x_0 = (-1, -1, 0)^T$:

$$x_k = \begin{pmatrix} k \\ \dfrac{1}{h} \\ 0 \end{pmatrix} ; \forall k \geq 1 \tag{5}$$

$$\lambda_1 = \frac{1}{h^2}; \quad \lambda_k = 0, \ \ \forall k \geq 2 \tag{6}$$

$\rightsquigarrow$ *We can not expect that this approximation converges to any solution.*

Another (related) problem is the meaning of $\lambda(t) \geq 0$. With $x_0 = (-1, -1, 0)^T$ λ has to be a distribution of degree 3 ($\dot{\delta}_0$) initially, so that the right-limit of $x(\cdot)$ satisfies $x_1(0^+) \geq 0$.

$\rightsquigarrow$ *writing* $\lambda \geq 0$ *is meaningless in general.*

We propose to embed (1) into a specific distributional inclusion inspired from Moreau's second order sweeping process in order to solve both issues.

- Mathematical formalism(s),
- Existence and uniqueness of solutions,
- Time-stepping discretization.

To begin with we perform a state space transformation with new state vector $z = Wx$, W square full-rank, and

$$z^T = (w, \dot{w}, \ddot{w}, ..., w^{(r-1)}, \xi^T) = (\bar{z}^T, \xi^T),\ \xi \in \mathbb{R}^{n-r} \tag{7}$$

$$\begin{cases} \dot{z}_1(t) = z_2(t) \\ \dot{z}_2(t) = z_3(t) \\ \vdots \\ \dot{z}_{r-1}(t) = z_r(t) \\ \dot{z}_r(t) = CA^r W^{-1} z(t) + CA^{r-1} B\lambda(t) \\ \dot{\xi}(t) = A_\xi \xi(t) + B_\xi z_1(t) \\ w(t) = z_1(t) \end{cases} \tag{8}$$

Since solutions may be distributions we rewrite the system as an equality of distributions

$$\begin{cases} Dz_1 = \{z_2\} + \nu_1 \\ Dz_2 = \{z_3\} + D\nu_1 + \nu_2 \\ Dz_3 = \{z_4\} + D^2\nu_1 + D\nu_2 + \nu_3 \\ \vdots \\ Dz_i = \{z_{i+1}\} + D^{(i-1)}\nu_1 + D^{(i-2)}\nu_2 + \ldots + D\nu_{i-1} + \nu_i \\ \vdots \\ Dz_{r-1} = \{z_r\} + D^{(r-2)}\nu_1 + \ldots + \nu_{r-1} \\ Dz_r = CA^r W^{-1}\{z\} + CA^{r-1}B\lambda. \end{cases} \tag{9}$$

and

$$\lambda = (CA^{r-1}B)^{-1}[D^{(r-1)}\nu_1 + \ldots + D\nu_{r-1}] + \nu_r \tag{10}$$

provided $CA^{r-1}B \neq 0$ (invertible in the case $m \geq 2$).

The higher order sweeping process characterizes the choice of the measures ν_i $(\langle \nu_i, \varphi \rangle = \int_I \varphi \, d\nu_i, \;\; \forall \varphi \in C_0^\infty(I))$ as follows:

$$d\nu_i \in -\partial \psi_{T_\Phi^{i-1}(\{Z_{i-1}\}(t^-))}(\{z_i\}(t^+)) \quad \text{on } [0, +\infty), \quad (1 \leq i \leq r)$$

where $Z_i = (z_1, z_2, ..., z_i), \quad (1 \leq i \leq r)$, and the sets $T_\Phi^i(\{Z_i\}(t^-))$ are tangent cones.

$$T_\Phi^0(Z_1) = \Phi, \;\; T_\Phi^1(Z_1) = T_\Phi(z_1), \;\; T_\Phi^2(Z_2) = T_{T_\Phi^1(Z_1)}(z_2),$$
$$T_\Phi^i(Z_i) = T_{T_\Phi^{i-1}(Z_{i-1})}(z_i).$$

In our case we always have

$$\Phi := \mathbb{R}^+$$

The similarity with the second order sweeping process

$$-d\nu \in \partial\psi_{V(q(t))}(\dot{q}(t^+))$$

is clear: we fit the cones together.

The main difference is that solutions of the higher order sweeping process are no longer measures but distributions of degree $\leq r$ where $r \geq 1$ is the relative degree between $w = Cx$ and λ.

Well-posedness: If $CA^{r-1}B > 0$ solutions exist on $[0, +\infty)$ satisfying $z_1(t) \geq 0$ and uniqueness holds in a specific set of distributions (generalization of Mechanics)

State jump mapping:

The following holds at all t:

$$\begin{array}{c} \{z_i\}(t^+) - \{z_i\}(t^-) \in -\partial\psi_{T_\Phi^{i-1}(\{Z_{i-1}\}(t^-))}(\{z_i\}(t^+)) \\ \Updownarrow \\ \{z_i\}(t^+) = \mathrm{prox}\left[T_\Phi^{i-1}(\{Z_{i-1}\}(t^-)); \{z_i\}(t^-)\right] \end{array} \tag{11}$$

and

$$\begin{array}{c} \{z_r\}(t^+) - \{z_r\}(t^-) \in -CA^{r-1}B\ \partial\psi_{T_\Phi^{r-1}(\{Z_{r-1}\}(t^-))}(\{z_r\}(t^+)) \\ \Updownarrow \\ \{z_r\}(t^+) = \mathrm{prox}_{(CA^{r-1}B)^{-1}}\left[T_\Phi^{r-1}(\{Z_{r-1}\}(t^-)); \{z_r\}(t^-)\right] \end{array} \tag{12}$$

The complementarity $0 \leq w(t) \perp \lambda \geq 0$ is now understood as

$$\boxed{0 \leq z_1(t^+) \perp d\nu_r(\{t\}) \geq 0}$$

Let us rewrite $d\nu_i = \chi_i(t)dt + d\mathcal{J}_i$, where $\chi_i(\cdot)$ is a function and $d\mathcal{J}_i$ is an atomic measure. Then

$$\begin{aligned} &\chi_i(t) = 0, \ \text{ a.e. } \ t \in [0, +\infty), \ \ (1 \leq i \leq r-1), \\ &\chi_r(t) \in -\partial \psi_{T_{\Phi}^{r-1}(\{z_1\}(t^-),\ldots,\{z_{r-1}\}(t^-))}(\{z_r\}(t^+)), \ \text{ a.e. } \ t \in [0, +\infty), \\ &0 \leq z_1(t^+) \perp \chi_r(t) \geq 0, \ \text{ a.e. } \ t \in [0, +\infty). \end{aligned} \tag{13}$$

Time-stepping algorithm

$$
\left\{
\begin{array}{l}
z_{i,k+1} - z_{i,k} - h z_{i+1,k+1} = \mu_{i,k+1} \in -\partial \psi_{T_{\Phi}^{i-1}(Z_{i-1,k})}(z_{i,k+1}) \;\; (1 \leq i \leq r-1). \\
z_{r,k+1} - z_{r,k} - hCA^{r}W^{-1} z_{k+1} = CA^{r-1}B \; \mu_{r,k+1} \\
\xi_{k+1} - \xi_{k} - hA_{\xi}\xi_{k+1} - hB_{\xi} z_{1,k+1} = 0 \\
\\
\mu_{r,k+1} \in -\partial \psi_{T_{\Phi}^{r-1}(Z_{r-1,k})}(z_{r,k+1}).
\end{array}
\right.
\tag{14}
$$

$\rightsquigarrow$ *The values of the measures* $dz_i((t_k, t_{k+1}])$ *and* $\mu_{i,k+1} \stackrel{\Delta}{=} d\nu_i((t_k, t_{k+1}])$ *are kept as primary variables.*

$\rightsquigarrow$ *This discretization does not try to approximate the distributional inclusion but a measure inclusion.*

Coming back to the third order example, the time-stepping HOSP solution is:

$$x_k = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} ; \forall k \geq 1 \tag{15}$$

$$\mu_{1,1} = 1; \quad \mu_{2,1} = 1, \tag{16}$$

$$\mu_{1,k} = 0, \mu_{2,k} = 0, \; \forall k \geq 2 \tag{17}$$

which converges to the time-continuous solution of the higher order Moreau's sweeping process, i.e. $x(0) = x_0, x(t) = (0, 0, 0)^T, \forall t > 0$.

For the case where the operator $d\nu \mapsto (z_1 \ \dots \ z_r)^T$ is positive real (dissipative), the following results can be proved:

- dissipation inequality:

$$\frac{1}{2} z_{k+1}^T J z_{k+1} - \frac{1}{2} z_k^T J z_k \leq -\frac{1}{2}(z_{k+1}-z_k)^T J (z_{k+1}-z_k) + h z_{k+1}^T J W A W^{-1} z_{k+1} \tag{18}$$

- Boundedness properties of discretized state and multiplier.
- local bounded variation of discretized state and multiplier.
- Convergence of discretized solutions towards some time functions.

All this borrows a lot from the proofs of Monteiro Marques for the second order sweeping process.

Differential Inclusions in Nonsmooth Mechanical Porblems: Shocks and Dry Friction, Birkhauser, 1993

Future works:

- Prove convergence to solutions of the HOSP (done only on particular cases).
- Nonlinear case: $\dot{x} = f(x) + g(x)\lambda$, $w = h(x)$.
- Nonautonomous case: $\dot{x} = Ax + B\lambda + Eu(t)$, $w = Cx + Fu(t)$.