Recent Advances on Complementarity Systems

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Topics

- Zeno states of complementarity systems

- Solution dependence on initial conditions

- Local observability (subsequently, joint extension with K. Çamlibel)

- Lyapunov stability (talk by K. Çamlibel)

References

• J.S. Pang and D.E. Stewart. Solution dependence on initial conditions in differential variational inequalities. Manuscript (August 2004).

• J.L. Shen and J.S. Pang. Linear complementarity systems: Zeno states. *SIAM Journal on Control and Optimization* (2005) to appear.

• J.L. Shen and J.S. Pang. Strongly regular differential variational systems. Manuscript (January 2005).

Consider the Nonlinear Complementarity System (NCS) of finding trajectories $x(t) \in \Re^n$ and $u(t) \in \Re^m$ satisfying

$$\boxed{\begin{array}{rcl} \dot{x}(t) & = & F(x(t), u(t)) \\ 0 \leq u(t) & \perp & H(x(t), u(t)) \geq 0 \end{array}}$$

where $F : \Re^{n+m} \rightarrow \Re^n$ and $H : \Re^{n+m} \rightarrow \Re^m$ are given **analytic** functions.

Blanket assumption on the state $x^0 \equiv x(t_0)$ at time t_0.

The nonlinear complementarity problem:

$$\boxed{0 \leq u \perp H(x^0, u) \geq 0}$$

has a *strongly regular* solution u^0.

Define 3 fundamental index sets corresponding to the pair (x^0, u^0):

$$\begin{array}{lcll} \alpha_0 & = & \{\, i \,:\, u_i^0 > 0 = H_i(x^0, u^0)\,\} & \text{the active } u\text{-indices} \\ \beta_0 & = & \{\, i \,:\, u_i^0 = 0 = H_i(x^0, u^0)\,\} & \text{the degenerate indices} \\ \gamma_0 & = & \{\, i \,:\, u_i^0 = 0 < H_i(x^0, u^0)\,\} & \text{the inactive } u\text{-indices.} \end{array}$$

Accordingly,

$$J_u H(x^0, u^0) \equiv$$

$$\left[\begin{array}{ccc} J_{u_{\alpha_0}} H_{\alpha_0}(x^0, u^0) & J_{u_{\beta_0}} H_{\alpha_0}(x^0, u^0) & J_{u_{\gamma_0}} H_{\alpha_0}(x^0, u^0) \\ J_{u_{\alpha_0}} H_{\beta_0}(x^0, u^0) & J_{u_{\beta_0}} H_{\beta_0}(x^0, u^0) & J_{u_{\gamma_0}} H_{\beta_0}(x^0, u^0) \\ J_{u_{\alpha_0}} H_{\gamma_0}(x^0, u^0) & J_{u_{\beta_0}} H_{\gamma_0}(x^0, u^0) & J_{u_{\gamma_0}} H_{\gamma_0}(x^0, u^0) \end{array} \right],$$

Strong regularity means

(a) the principal submatrix $J_{u_{\alpha_0}} H_{\alpha_0}(x^0, u^0)$ is nonsingular, and

(b) the Schur complement, denoted D,

$$J_{u_{\beta_0}} H_{\beta_0}(x^0, u^0) - J_{u_{\alpha_0}} H_{\beta_0}(x^0, u^0) \left[J_{u_{\alpha_0}} H_{\alpha_0}(x^0, u^0) \right]^{-1} J_{u_{\beta_0}} H_{\alpha_0}(x^0, u^0)$$

is a P-matrix (i.e., all its principal minors are positive).

Hence, $\exists$ neighborhoods $\mathcal{U}_0$ of u^0 and $\mathcal{V}_0$ of x^0, and a piecewise smooth function $u : \mathcal{V}_0 \to \mathcal{U}_0$ such that for every $x \in \mathcal{V}_0$, $u(x)$ is the only vector u in $\mathcal{U}_0$ satisfying the NCP:

$$0 \leq u \perp H(x, u) \geq 0.$$

Consequently, the NCS is equivalent to

$$\dot{x} = F(x, u(x)),$$

where the right-hand function is piecewise smooth in x near x^0.

Hence, near t_0, the NCS has a unique C^1 solution trajectory $x^*(t)$, which induces a piecewise smooth trajectory $u^*(t) \equiv u(x^*(t))$.

Define

$$\begin{aligned}
\alpha(t) &= \{\, i \,:\, u_i^*(t) > 0 = H_i(x^*(t), u^*(t)) \,\} \\
\beta(t) &= \{\, i \,:\, u_i^*(t) = 0 = H_i(x^*(t), u^*(t)) \,\} \\
\gamma(t) &= \{\, i \,:\, u_i^*(t) = 0 < H_i(x^*(t), u^*(t)) \,\}.
\end{aligned}$$

A Strong Non-Zenoness Theorem.

There exist a scalar $\varepsilon \in (0, \varepsilon_0)$ and two triples of index sets, $(\alpha_+, \beta_+, \gamma_+)$ and $(\alpha_-, \beta_-, \gamma_-)$ such that

$$\begin{aligned}
(\,\alpha(t), \beta(t), \gamma(t)\,) &= (\alpha_-, \beta_-, \gamma_-\,), \quad \forall\, t \in [\, t_0 - \varepsilon, t_0\,) \\
(\,\alpha(t), \beta(t), \gamma(t)\,) &= (\alpha_+, \beta_+, \gamma_+\,), \quad \forall\, t \in (\, t_0, t_0 + \varepsilon\,].
\end{aligned}$$

A piecewise analytic corollary.

If $(x^*(t), u^*(t))$ is a solution trajectory of the NCS on the interval $[a, b]$ such that $u^*(t)$ is a strongly regular solution of the NCP: $0 \leq u \perp H(x^*(t), u) \geq 0$, then $\exists$ finitely many subintervals

$$a = t_0 < t_1 < \cdots < t_{N-1} < t_N = b,$$

the functions $(x^*(t), u^*(t))$ are analytic at all $t \in (t_{i-1}, t_i)$, for $i = 1, \ldots, N$.

Picard iterations and Lie Derivatives

Let $f : \Re^n \rightarrow \Re^n$ and $C : \Re^n \rightarrow \Re^m$ be analytic functions defined in a neighborhood of $x^* \in \Re^n$. Define the sequences of functions $\{\widehat{x}^k(t)\}_{k=0}^{\infty}$ and $\{L_f^k C(x)\}_{k=0}^{\infty}$ iteratively by:

$$
\boxed{
\begin{aligned}
\widehat{x}^0(t) \;&\equiv\; x^*, \\
\widehat{x}^k(t) \;&\equiv\; x^* + \int_0^t f(\,\widehat{x}^{k-1}(s)\,)ds, \quad k \geq 1
\end{aligned}
}
$$

$$
\boxed{
\begin{aligned}
L_f^0 C(x) \;&\equiv\; C(x), \\
L_f^k C(x) \;&\equiv\; (\,J L_f^{k-1} C(x)\,) f(x), \quad k \geq 1,
\end{aligned}
}
$$

where Jg denotes the Jacobian of g

All the functions $\widehat{x}^k(t)$ and $L_f^k C(x)$ are analytic functions.

They are connected via the equality:

$$\left.\frac{d^i C(\widehat{x}^k(t))}{dt^i}\right|_{t=0} = L_f^i C(x^*), \ \ \forall i = 1, \ldots, k,$$

yielding the expansion

$$C(\widehat{x}^k(t)) = \sum_{j=0}^{k} L_f^j C(x^*) \frac{t^j}{j!} + O(t^{k+1}).$$

Returning to the NCS, assume that $u^0 = 0 = H(x^0, 0)$. Let

$$f(x) \equiv F(x, 0) \text{ and } C(x) \equiv H(x, 0).$$

Expansion of solution near a strongly regular state.

If $L_f^i C(x^0) = 0$ for all $i = 0, \cdots, k-1$, then

$$\boxed{\begin{aligned} x^*(t) &= \widehat{x}^{k+1}(t) + \frac{t^{k+1}}{(k+1)!} J_u F(x^0, u^0) v^k + o(t^{k+1}) \\ u^*(t) &= \frac{t^k}{k!} v^k + o(t^k), \end{aligned}}$$

where v^k is the unique solution of the LCP:

$$0 \leq v \perp L_f^i C(x^0) + J_u H(x^0, 0) v \geq 0.$$

Dependence on initial conditions. First, for the LCS:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ 0 \leq u(t) &\perp Cx(t) + Du(t) \geq 0 \\ x(0) &= \xi. \end{aligned}$$

Assumption. $B\mathrm{SOL}(Cx, D)$ is a singleton $\forall\ x \in \Re^n$.

By classical ODE theory,

• $\exists!$ a C^1 (in t) trajectory $x(t,\xi)$ for every $\xi \in \Re^n$; moreover $x(t,\cdot)$ is Lipschitz continuous in initial condition for every $t > 0$.

More interestingly, $x(t,\cdot)$ is semismooth, thus, directionally differentiable.

Its directional derivative

$$x'_\xi(t,\xi;\eta) \equiv \lim_{\tau\downarrow 0} \frac{x(t,\xi+\tau\eta) - x(t,\xi)}{\tau}$$

- is the unique solution of a variational equation,
- satisfies the limit condition, semismoothness property:

$$\lim_{\xi^0 \neq \xi \to \xi^0} \frac{x'_\xi(t,\xi;\xi-\xi^0) - x'_\xi(t,\xi^0;\xi-\xi^0)}{\| \xi - \xi^0 \|} = 0.$$

Application to boundary-value problems

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ 0 \leq u(t) &\perp Cx(t) + Du(t) \geq 0 \\ b &= Mx(0) + Nx(T). \end{aligned}$$

Reformulation as a semismooth equation:

$$F(\xi) \equiv M\xi + Nx(T,\xi) - b = 0.$$

- Existence, uniqueness, and Lipschitz continuity of a solution $\xi(b)$, via an implicit function theorem for semismooth functions.

- Numerical computation of $\xi(b)$ via a *nonsmooth Newton method*.

Extension to DVI

$$\begin{aligned} \dot{x} &= f(x,u) \\ u &\in \text{SOL}(K, F(x,\cdot)) \\ x(0) &= \xi. \end{aligned}$$

Assumptions.

- f and F are C^1 near (ξ^0, u^0)

- $u^0 \in \text{SOL}(K, F(\xi^0, \cdot))$ is strongly regular

- K is polyhedral (or more generally, is such that the Euclidean projector onto K is semismooth).

The solution map $x(t, \cdot)$ has the same semismooth property at ξ^0; moreover, the directional derivative $x'_\xi(t, \xi^0; \eta)$ can be obtained by "directionally differentiating" the right-hand side of ODE

$$\dot{x} = f(x, u(x))$$

at ξ^0 along the direction η.

Local observability

The state $\xi^0 \in \Re^n$ is said to be *T-time locally observable* for the DVI with respect to the output function $\Phi : \Re^n \to \Re^p$, where $T \in (0, \infty)$, if there exists a neighborhood $\mathcal{N}_0$ of ξ^0 such that

$$\left.\begin{array}{rcl} \Phi(x(t,\xi)) & = & \Phi(x(t,\xi^0)), \ \forall\, t \in [\,0, T\,] \\ \xi & \in & \mathcal{N}_0 \end{array}\right\} \Rightarrow \ \xi = \xi^0.$$

If there exists a scalar $\varepsilon_0 > 0$ such that ξ^0 is ε-time locally observable for the DVI with respect to the function Φ for all $\varepsilon \in (0, \varepsilon_0]$, then we say that ξ^0 is *short-time locally observable*.

Theorem. Let $\Psi, \Phi : \Re^n \to \Re^n$ be B-differentiable at $\xi^0 \in \Re^n$. Let $x(t,\xi)$ for $(t,\xi) \in [0,T] \times \mathcal{N}$ be a solution trajectory of:

$$\text{ODE: } \dot{x} = \Psi(x), \quad x(0) = \xi,$$

where $T > 0$ is a suitable scalar and $\mathcal{N}$ is a suitable neighborhood of ξ^0. If $\varepsilon_0 \in (0,T]$ exists such that for all $\varepsilon \in (0,\varepsilon_0]$,

$$\boxed{\left\{ \Phi'(x(t,\xi^0); x'_\xi(t,\xi^0;\eta)) = 0, \ \forall\, t \in [0,\varepsilon] \right\} \Rightarrow \eta = 0,}$$

then ξ^0 is T'-time locally observable for the ODE with respect to Φ for any $T' \in (0,T]$.

For the LCS with a linear output function, the sufficient condition becomes the unique solvability of finitely many "semi-inifinite", homogeneous, linear inequality systems.

Concluding Remarks. We have

• discussed Zenoness + mode switches, via an expansion lemma;

• established semismoothness of solution operator and explained its importance;

• very briefly discussed the issue of local observability.