The Kinematics of Unilaterality

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A finite freedom system

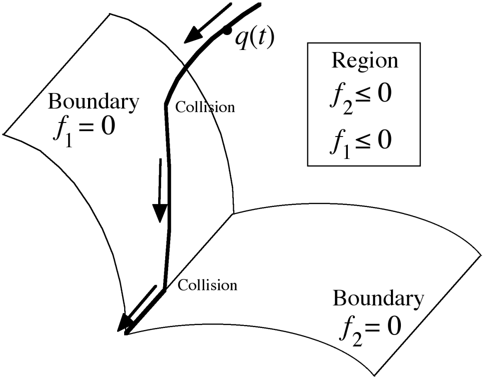
The evolution

$$t \mapsto q := (q^1, \ldots, q^n)$$

is required to comply, for every t in some interval I, with

$$f(t, q) \leq 0$$

(a single inequality to begin with).



In other words, the *moving point* $q(t)$ is required to belong at every time t to the *moving set*

$$\Phi(t) := \{x \in \mathbb{R}^n \mid f(t, x) \leq 0\}.$$

It is assumed that, for $t \in I$ and $x \in \mathbb{R}^n$, the *gradient* $\nabla f(t,x) := (\partial f/\partial x^1, \ldots, \partial f/\partial x^n)$ is a nonzero n-vector.

Let t be such that the *right-side derivative* $q'^{+}(t)$, (the *right-side velocity*) exists.

Through the chain rule, the function $\tau \mapsto f(\tau, q(\tau))$ possesses at $\tau = t$ a right-side derivative equal to $f'_t(t, q(t)) + q'^{+}(t).\nabla f(t, q(t))$.

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This should be ≤ 0 if the inequality
is satisfied at t as *equality*.

If inequality holds strictly at t, no sign condition
comes to restrain right-side derivatives.

For $t \in I$ and $x \in \mathbb{R}^n$, put

$$\Gamma(t,x) := \begin{cases} \{v \in \mathbb{R}^n \mid f'_t(t,x) + v.\nabla f(t,x) \leq 0\} \\ \qquad\qquad\qquad\qquad \text{if} \quad f(t,x) \geq 0 \\ \mathbb{R}^n \quad \text{otherwise} \end{cases}$$

so that the precedings means $q'^{+}(t) \in \Gamma(t, q(t))$.

Here is a converse

Assume that I, nonnecessarily compact, contains its origin t_0 and that $q : I \longrightarrow \mathbb{R}^n$ is *locally absolutely continuous*.

Equivalently the (two-side) derivative dq/dt exists a.e. in I and equals some $u \in \mathcal{L}^1_{\mathrm{loc}}(I; \mathbf{R}^n)$ with

$$\forall t \in I : \quad q(t) = q(t_0) + \int_{t_0}^{t} u(s)\, ds.$$

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Viability Lemma. *Assume in addition that*

$$\frac{dq}{dt} \in \Gamma(t, q(t)) \quad \text{a.e. in } I.$$

If $f(t, q(t)) \leq 0$ *is verified at* t_0*, then it is verified for every subsequent* t.

A condition of the above form is called
a *differential inclusion*.

By a *selector* of the *multifunction* $(t, x) \mapsto \Gamma(t, x)$,
one means a single-valued function
say $(t, x) \mapsto \gamma(t, x)$,
such that $\gamma(t, x) \in \Gamma(t, x)$ for every t and x.

Then

$$\frac{dq}{dt} = \gamma(t, q(t))$$

is a *differential equation*

whose (locally absolutely continuous) solutions,

if any,

consequent to some *initial condition*

verifying $q(t_0)) \in \Phi(t_0)$,

meet the assumptions of the Viability Lemma,

making $q(t)$ ***belong to*** $\Phi(t)$

for every subsequent t.

Basic example: the "lazy selector".

Define $\gamma(t, x)$ as *the element of minimal Euclidean norm* in $\Gamma(t, x)$.

Then a solution to $dq/dt = \gamma(t, q(t))$ consequent to some initial position $q(t_0)$ in $\Phi(t_0)$ may be described as follows.

The point $q(t)$ belongs for every t
to the moving region $\Phi(t)$.

So long as it lies in the interior of $\Phi(t)$,
q *stays at rest*.

It is only if the boundary of $\Phi(t)$,
i.e. the hypersurface $f(t,.) = 0$,
moves inward and reaches q
that this point takes on a velocity
in inward normal direction,

so as to go on belonging to $\Phi(t)$.

We have proposed to call *Sweeping Process* this way of associating some point motions to the given motion of a set (in $\mathbb{R}^n$ or in a real Hilbert space).

If, at time t, some point x lies on the hypersurface

$$f(t, .) = 0,$$

the vector $\nabla f(t, x)$ is normal to this hypersurface

and directed outward of $\Phi(t)$.

The half-line emanating from the origin of $\mathbb{R}^n$,

generated by $\nabla f(t, x)$, constitutes

the (outward) *normal cone* to $\Phi(t)$ at point x.

Notation: $\mathrm{N}_{\Phi(t)}(x)$.

For x in the interior of $\Phi(t)$,
it proves consistent to view $\mathrm{N}_{\Phi(t)}(x)$
as reduced to the *zero* of $\mathbb{R}^n$,
while the cone shall be defined as *empty* if $x \notin \Phi(t)$.

By discussing the various cases occuring in the calculation of $\gamma(t,x)$, one sees that every solution q to

$$\frac{dq}{dt} = \gamma(t, q(t))$$

verifies, for almost every t, the differential inclusion

$$-\frac{dq}{dt} \in \mathrm{N}_{\Phi(t)}(q(t)).$$

Unexpectedly *the converse is true*, i.e. the above in spite of its right-hand side being multivalued is *equivalent* to the differential equation,

as far as locally absolutely continuous solutions are concerned.

Proof: Let q be a solution to $-dq/dt \in \mathrm{N}_{\Phi(t)}(q)$.

For almost every t, the *two-side* derivative $q' = dq/dt$ exists.

Therefore $\mathrm{N}_{\Phi(t)}(q) \neq \emptyset$ hence $q(t) \in \Phi(t)$ and the same for every t, by continuity.

- For t such that $q(t) \in \mathrm{interior}\ \Phi(t)$, inclusion implies $q' = 0$, so trivially $dq/dt = \gamma(t, q(t))$.

- Otherwise, suppose $q(t) \in \mathrm{boundary}\ \Phi(t)$, i.e. function $\tau \mapsto f(\tau, q(\tau))$ vanishes at $\tau = t$.

  Then the right-derivative

  $f'_t(t, q(t)) + q'^{+}(t).\nabla f(t, q(t))$, if it exists, is ≤ 0 while, symmetrically, the left-derivative is ≥ 0.

Therefore $q'(t)$, when it exists, satisfies

$$f'_t(t, q(t)) + q'(t).\nabla f(t, q(t)) = 0,$$

i.e. it belongs to the boundary of the half-space $\Gamma(t, q(t))$.
Furthermore, the inclusion entails that $q'(t)$ is directed along the inward normal to the half-space.

All this elementarily characterizes $q'(t)$
as the *proximal point* to 0 in $\Gamma(t, q(t))$
namely $\gamma(t, q(t))$.

It was under the formulation $-dq/dt \in \mathrm{N}_{\Phi(t)}(q)$ that the *Sweeping Process* was primitively introduced, with $\Phi(t)$ denoting a nonempty *closed convex*, nonnecessarily smooth, subset of a real Hilbert space $\boldsymbol{H}$.

The motivation then was in the
quasi-static evolution of elastoplastic systems.

The convexity assumption allows one to establish
the *existence* of solutions
under rather mild conditions concerning the evolution
of $\Phi(t)$, even *discontinuous*.

Another consequence of the convexity of Φ is that the multifunction $x \mapsto \mathrm{N}_{\Phi(t)}(x)$ is *monotone* in the following sense:

Whichever are x_1, x_2 in $\boldsymbol{H}$,
y_1 in $\mathrm{N}_{\Phi(t)}(x_1)$, y_2 in $\mathrm{N}_{\Phi(t)}(x_2)$,
one has $(x_1 - x_2).(y_1 - y_2) \geq 0$,
the dot denoting the scalar product of $\boldsymbol{H}$.

By elementary calculation, this property entails that, if $t \mapsto q_1(t)$ and $t \mapsto q_2(t)$ are two solutions, the distance $\|q_1 - q_2\|$ is a *non-increasing* function of t.

It follows that *at most one solution*
can agree with some initial position $q(t_0)$.

Another source of interest of the formulation

$$-\frac{dq}{dt} \in \mathrm{N}_{\Phi(t)}(q)$$

is to render evident that

the successive positions of the point q
are connected with those of the moving set Φ
in a *rate-independent* way.

In fact, because the right-hand member is a *cone*,
the differential inclusion is found invariant under any
non-decreasing differentiable change of variable.

Implicit versus *explicit* time-stepping

Let $[t_\mathrm{i}, t_\mathrm{f}]$, with length h, be a time-step.
From an estimate q_i of $q(t_\mathrm{i})$,
resulting from the antecedent time-step,
computation has to deliver an estimate q_f of $q(t_\mathrm{f})$.

The first formulation

$$\frac{dq}{dt} = \gamma(t, q(t))$$

induces one to take $u_\mathrm{i} = \gamma(t_\mathrm{i}, q_\mathrm{i})$ as an estimate
of the velocity, so generating the prediction

$$q_\mathrm{f} = q_\mathrm{i} + h u_\mathrm{i}$$

a computation scheme of the *explicit* type.

If the second formulation

$$-\frac{dq}{dt} \in \mathrm{N}_{\Phi(t)}(q(t))$$

is discretized by viewing $(q_{\mathrm{f}} - q_{\mathrm{i}})/h$ as a representative of the velocity, a strategy of the explicit type would not allow one to express q_{f}, since the right-hand member is multivalued.

In contrast, the *implicit* strategy consists in invoking the value that this right-hand member would take at the *unknown* point q_{f}

so one has to solve

$$q_{\mathrm{i}} - q_{\mathrm{f}} \in \mathrm{N}_{\Phi(t_{\mathrm{f}})}(q_{\mathrm{f}})$$

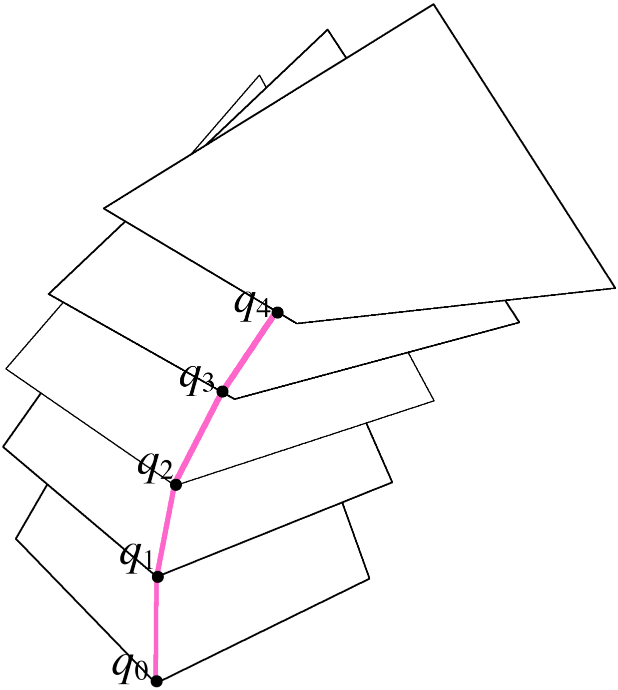
(the positive factor h has been dropped since $\mathrm{N}_{\Phi(t_{\mathrm{f}})}$ is a cone).

This inclusion qualifies q_{f} as
an *orthogonal projection* of q_{i}
onto the closed set $\Phi(t_{\mathrm{f}})$.

In case $\Phi(t_{\mathrm{f}})$ is *convex*, the projection is unique
and q_{f} equals the *nearest point* to q_{i} in $\Phi(t_{\mathrm{f}})$.

One calls this the *catching-up algorithm*.

Catching up



Complementarity

Let q, associated with u by $q(t) = q(t_0) + \int_{t_0}^{t} u(s)\, ds$, verify

$$-dq/dt \in \mathrm{N}_\Phi(q) \quad \text{a.e. in } I. \tag{1}$$

Let $t_1 \in I$ such that u possesses a limit on the right of t_1, say u_1^+ $(= \dot{q}^+(t_1))$.

If $f_1 := f(t_1, q(t_1)) = 0$, it was seen that

$$\dot{f}_1^+ = f_t'(t_1, q(t_1)) + u_1^+ . \nabla f(t_1, q(t_1)) \leq 0.$$

(1) means the existence of $t \mapsto \lambda(t) \leq 0$ such that $u(t) = \lambda(t) \nabla f(t, q(t))$.

Since ∇f is continuous and nonzero, the assumed existence of u_1^+ secures that of the right-limit λ_1^+ and

$$u_1^+ = \lambda_1^+ \nabla f(t_1, q(t_1)).$$

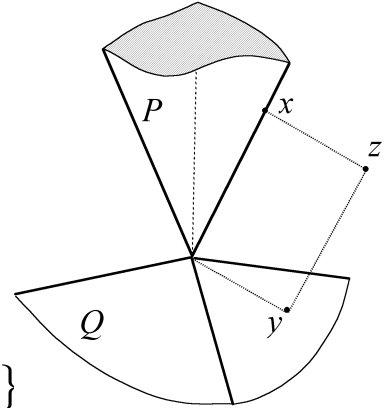
If $\dot{f}_1^+ < 0$, instant t_1 is followed by an interval throughout which $f < 0$. This implies $u = 0$, so that λ vanishes on this interval and consequently $\lambda_1^+ = 0$.

Summing up, one has

$$\dot{f}_1^+ \leq 0, \quad \lambda_1^+ \leq 0, \quad \dot{f}_1^+ \lambda_1^+ = 0,$$

a system of *complementarity conditions*.

An example of complementarity



P: a closed convex cone in H (a Hilbert space).

By definition, its *polar cone* is

$$P^\star = \{u \in H | \forall u \in P : u.v \leq 0\}$$

Symmetrically , P equals the polar cone of Q

The Dual Cone Lemma:

The two following statements are *equivalent*

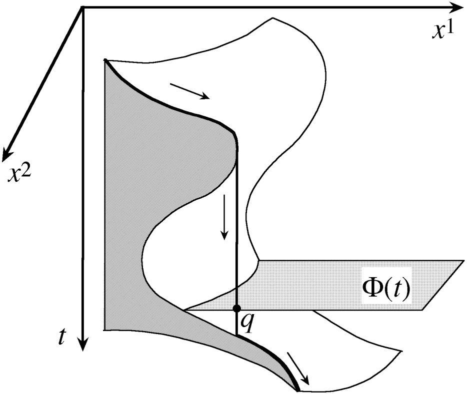
$$z = x + y, \quad x \in P, \quad y \in Q, \quad x.y = 0 \qquad (1)$$

$$x = \mathrm{prox}\,(P, z), \quad y = \mathrm{prox}\,(Q, z) \qquad (2)$$

A hydromechanical image

Let (t, x_1, x_2) be orthonormal axes in physical space, with t directed *downward*.

$\Phi(t)$ is the section at level t of a *ground cavity*.



The graph of a solution $t \mapsto q(t) \in \mathbb{R}^2$ to

$$-\frac{dq}{dt} \in \mathrm{N}_{\Phi(t)}(q(t))$$

may be visualized as a falling *stream of water*.

When flowing on a part of the wall *oriented upward*, water follows a *line of steepest descent*.
When crossing the rim of a possible overhang, water *gets loose* from the wall and *falls freely* into the cavity.

Here, under the complication added by *unilaterality*, the comparison of the two formulations of the sweeping process merely reflects the equivalence of the two standard properties of the lines of steepest descent in a surface: at each point on such a line

- the slope is maximal,
- the direction is orthogonal to the level curve of the surface.

All what precedes adapts to the case where $\Phi(t)$ is defined by a set of inequalities

$$f_\alpha(t,q) \leq 0, \quad \alpha = 1, 2, \ldots, \kappa$$

Normal cones are then polyhedral,
with variable number of edges.

Coming to Dynamics

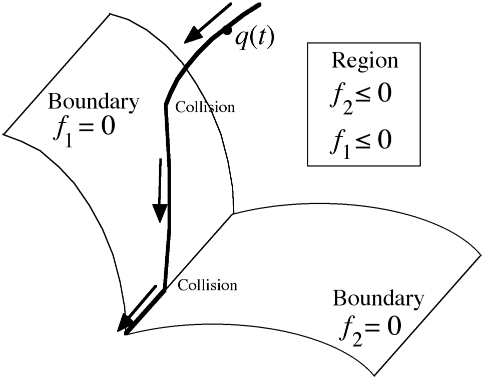
The *position* function

$$t \mapsto q := (q^1, \ldots, q^n)$$

and the *velocity* function

$$t \mapsto u := (u^1, \ldots, u^n)$$

are related through



$$q(t) = q(t_0) + \int_{t_0}^{t} u(\tau)\, d\tau$$

(or through more elaborate kinematical relationships like this: If a member of the system is a *rigid body* $\mathcal{B}$, one considers the components of its *spin vector* relative to an orthonormal frame made of *principal axes of inertia* at the mass center).

The basic example of a single particle in Euclidean space

Equation of Dynamics

$$m\frac{du}{dt} = e(t,q,u) + r$$

e: given force,

r: the total *constraint force*

The description of a constraint *in Mechanics* requires fondamentally *more* than giving the *geometric* restriction it imposes to the system position.

Some information about the constraint *realization* should be provided, in terms of constraint forces.

Non-adhesive frictionless confinement

of a particle by a material boundary.

The permitted region:

$$\Phi(t) = \{x \in \mathbb{E} : f(t,x) \leq 0\}$$

Stipulations about the constraint force r :

- if $f(t,q) < 0$, then $r = 0$ (contact process)
- if $f(t,q) = 0$, then (no friction, no adhesion)

$$\exists \lambda \geq 0 \text{ such that } r = -\lambda \, \nabla f(t,q)$$

This summarizes into

$$-r \in \mathrm{N}_{\Phi(t)}(q)$$

which also involves $q \in \Phi(t)$ (otherwise $\mathrm{N}_{\Phi(t)}(q) = \emptyset$)

The decisive move in the "Contact Dynamics" approach

As before, put

$$\Gamma(t,x) := \begin{cases} \{v \in \mathbb{R}^n \mid f'_t(t,x) + v.\nabla f(t,x) \leq 0\} \\ \qquad\qquad\qquad\qquad \text{if} \quad f(t,x) \geq 0 \\ \mathbb{R}^n \quad \text{otherwise} \end{cases}$$

The writing

$$-r \in \mathrm{N}_{\Gamma(t,q)}(u)$$

is found to imply $-r \in \mathrm{N}_{\Phi(t)}(q)$ and in addition

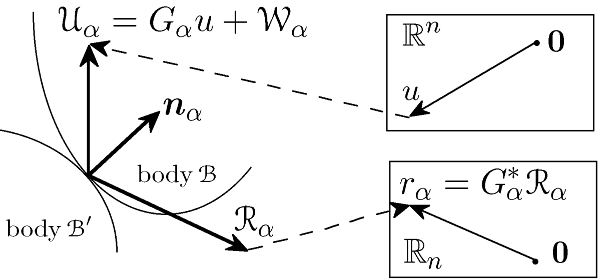
$$u \in \Gamma(t,q)$$

which allows one to invoque the *Viability Lemma*.

Local and global variables

From t, q, u, one expresses the *relative velocity* at the contact labelled α

$\mathcal{U}_\alpha = G_\alpha u + \mathcal{W}_\alpha \in \mathbb{E}^3$



The corresponding *contact force* $\mathcal{R}_\alpha \in \mathbb{E}^3$ has generalized components

$$r_\alpha = G^*_\alpha \mathcal{R}_\alpha \in \mathbb{R}_n$$

The linear mappings (depending on q and t)

$G_\alpha : \mathbb{R}^n \to \mathbb{E}^3$ and $G^*_\alpha : \mathbb{E}^3 \to \mathbb{R}_n$

are *transpose to each other.*

In numerical computation, as well as in mathematical studies, one agrees to extend, locally, the definitions of $\boldsymbol{n}_\alpha$, G_α, $\mathcal{W}_\alpha$ etc. to situations where $f_\alpha(t,q) \neq 0$.

Let $g_\alpha(t,q)$ denote the *gap* at the possible contact labelled α (counted negative in case of overlap).

Classically

$$\frac{dg_\alpha}{dt} = \boldsymbol{n}_\alpha . \mathcal{U}_\alpha$$

Define

$$\mathcal{K}_\alpha(t,q) := \begin{cases} \{\mathcal{V} \in \mathbf{E}^3 \mid \mathcal{V}.\boldsymbol{n}_\alpha \geq 0\} & \text{if} \quad g_\alpha(t,q) \leq 0 \\ \mathbf{E}^3 \quad \text{otherwise.} \end{cases}$$

This is the set of the values of the local *right-velocity* of $\mathcal{B}$ relatively to $\mathcal{B}'$ (the latter may be a member of the system or an external obstacle with prescribed motion) which are *compatible with non-interpenetration*.

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By applying the *Viability Lemma* to inequality $g_\alpha(t,q) \geq 0$, one obtains:

If $\mathcal{U}_\alpha \in \mathcal{K}_\alpha$ a.e. in the time interval
and if $g_\alpha \geq 0$ holds at initial instant,
then it holds on the whole interval.

A package of information concerning the contact force $\mathcal{R}_\alpha \in \mathbb{E}_3$ is called *a contact law*

Assume laws of the form

$$\mathrm{law}_\alpha(t, q, \mathcal{U}_\alpha, \mathcal{R}_\alpha) = \mathrm{true}$$

with *contact velocity* $\mathcal{U}_\alpha = G_\alpha u + \mathcal{W}_\alpha$ as before.

A contact law is said *prospective* if, among other phenomenological stipulations, it involves

- in all cases $\mathcal{U}_\alpha \in \mathcal{K}_\alpha$,
- if $\mathcal{U}_\alpha \in \mathrm{interior}\, \mathcal{K}_\alpha$, then $\mathcal{R}^\alpha = 0$.

In other words, it involves the implications

$$g_\alpha(t, q) \leq 0 \;\Rightarrow\; \boldsymbol{n}_\alpha . \mathcal{U}_\alpha \geq 0$$

$$\boldsymbol{n}_\alpha . \mathcal{U}_\alpha > 0 \;\Rightarrow\; \mathcal{R}_\alpha = 0.$$

Such a contact law secures $g_\alpha \geq 0$
provided the latter holds at initial instant.

Giving *Coulomb's friction* the form of a prospective contact law is only the matter of writing code adequately.

But what follows introduces some theoretical consistency

Let $\mathbf{T}$ be the orthogonal space to $\boldsymbol{n}$ in $\mathbf{E}^3$ and

$$\mathcal{R} = \mathcal{R}_{\rm T} + \mathcal{R}_{\rm N}\,\boldsymbol{n}, \quad \mathcal{R}_{\rm T} \in \mathbf{T}, \quad \mathcal{R}_{\rm N} \in \mathbf{R},$$

$$\mathcal{U} = \mathcal{U}_{\rm T} + \mathcal{U}_{\rm N}\,\boldsymbol{n}, \quad \mathcal{U}_{\rm T} \in \mathbf{T}, \quad \mathcal{U}_{\rm N} \in \mathbf{R}.$$

Let $D_1 := \{\mathcal{R}_{\rm T} \in \mathbf{T} \mid \mathcal{R}_{\rm T} + \mathbf{n} \in \mathcal{C}\}$, the 'unit section' of the *Coulomb cone* $\mathcal{C}$. Define in $\mathbf{T}$ the real function

$$\mathcal{T} \in \mathbf{T} \mapsto \varphi_1(\mathcal{T}) := \sup\{\mathcal{S}.\mathcal{T} \mid \mathcal{S} \in -D_1\}.$$

(the 'dissipation function'). In the traditional case of *isotropic* friction, one simply has $\varphi_1(\mathcal{T}) = \gamma\|\mathcal{T}\|, \gamma > 0$.

The Coulomb cone depends on t and q.
Put $\mathcal{C} = \{\mathbf{0}\}$ in case of no-contact.

Using arguments from Convex Analysis, G. De Saxcé established that the system of conditions

$$\mathcal{U} \in \mathcal{K},\ \mathcal{R} \in \mathcal{C},\ -\mathcal{U}.\mathcal{R} = \varphi_1(\mathcal{U}_{\mathrm{T}})\mathcal{R}_{\mathrm{N}} \qquad (1)$$

makes a *prospective contact law* which, in the standard situation, reduces to the law of Coulomb.

Furthermore

$$\forall\, \mathcal{V} \in \mathcal{K}, \quad \forall\, \mathcal{S} \in \mathcal{C} : \quad \mathcal{V}.\mathcal{S} + \varphi_1(\mathcal{V}_{\mathrm{T}})\,\mathcal{S}_{\mathrm{N}} \geq 0$$

i.e. (1) expresses that the real function $(\mathcal{V}, \mathcal{S}) \mapsto \mathcal{V}.\mathcal{S} + \varphi_1(\mathcal{V}_{\mathrm{T}})\,\mathcal{S}_{\mathrm{N}}$, separately convex in $\mathcal{V}$ and $\mathcal{S}$, attains at point $(\mathcal{U}, \mathcal{R})$ its *minimal value* relative to the product set $\mathcal{K} \times \mathcal{C}$ and that *this minimal value is zero*.

A success of Coulomb's law

WALL SUPPORTED BY DEFORMING GROUND

105 blocks
Block height : 0.049 m
Block widths : 0.124 m *and* 0.062 m

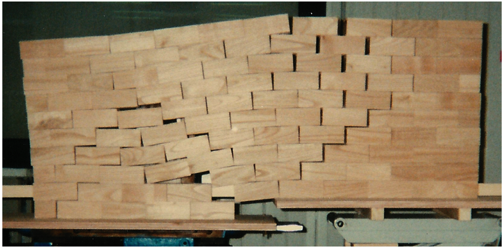
Right foundation is fixed.
Left foundation moves down.
Final displacement 0.06 m

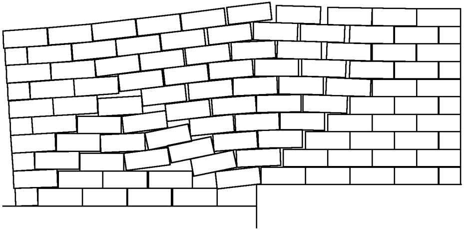
Experiment:

M. Jean,
Laboratoire de Mécanique et d'Acoustique
(UPR CNRS 7051), Marseille.

Numerical simulation by the Contact Dynamics *method:*

J. J. Moreau,
Laboratoire de Mécanique et Génie Civil
(UMR CNRS 5508), Université Montpellier II.





Rigid body Collisions

don't possess fully satisfactory models.

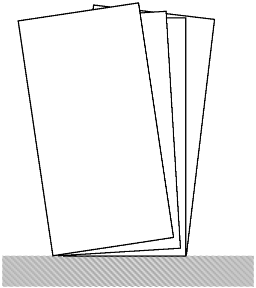
Denote by "$-$" and "$+$" the limits on the left and on the right of a collision instant.

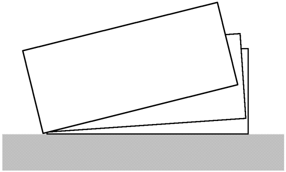
Newton's *restitution* law for a *single collision*

$$\mathcal{U}_{\mathrm{N}}^{+} = -e\,\mathcal{U}_{\mathrm{N}}^{-}, \quad 0 \le e \le 1$$

doesn't yield plausible results for *multicontact* systems.

To rock or not to rock...





An efficient trick

Use a *prospective contact law* (e.g. Coulomb in prospective form) to connect every contact *percussion* $\mathcal{P}_\alpha$ to a *weighted mean* $\mathcal{U}^a_\alpha$ of $\mathcal{U}^-_\alpha$ and $\mathcal{U}^+_\alpha$

$$\mathcal{U}^a_{\alpha\mathrm{N}} = \frac{\rho_\alpha}{1+\rho_\alpha}\mathcal{U}^-_{\alpha\mathrm{N}} + \frac{1}{1+\rho_\alpha}\mathcal{U}^+_{\alpha\mathrm{N}} \qquad (1)$$

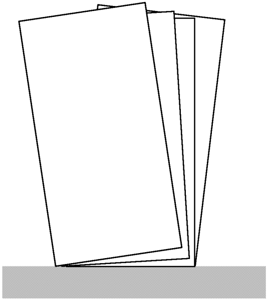
$$\mathcal{U}^a_{\alpha\mathrm{T}} = \frac{\tau_\alpha}{1+\tau_\alpha}\mathcal{U}^-_{\alpha\mathrm{T}} + \frac{1}{1+\tau_\alpha}\mathcal{U}^+_{\alpha\mathrm{T}}. \qquad (2)$$

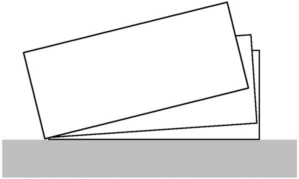
The empirical parameters ρ_α and τ_α will be called *the normal coefficient of restitution* and *the tangential coefficient of restitution* at the contact labelled α, denominations justified by what follows.

That the law is *prospective* implies that $\mathcal{P}^\alpha$ can be nonzero only if $\mathcal{U}^a_{\alpha \mathrm{N}} = 0$, i.e. $\mathcal{U}^+_{\alpha \mathrm{N}} = -\rho_\alpha \mathcal{U}^-_{\alpha \mathrm{N}}$, which formally is Newton's restitution ($0 \leq \rho_\alpha \leq 1$). But the above formulation also allows $\mathcal{P}^\alpha = 0$, in which case only inequality $\mathcal{U}^a_{\alpha \mathrm{N}} \geq 0$ is asserted.

It is the global calculation, involving all contacts together through the equation of dynamics, which decides between these two alternatives.

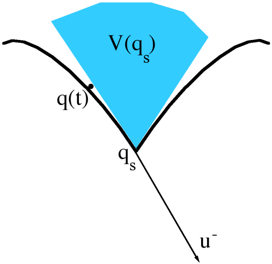
Similarly, the global calculation, if friction is large enough, may end in the *zero sliding* case of Coulomb's law at contact α. Then $\mathcal{U}^+_{\alpha \mathrm{T}} = -\tau_\alpha \mathcal{U}^-_{\alpha \mathrm{T}}$, which is *tangential restitution*.

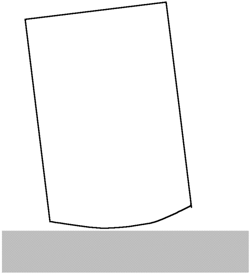


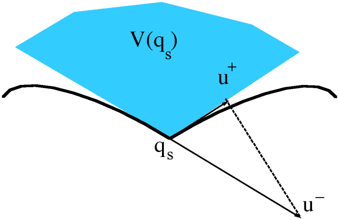


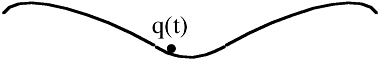
To rock or not to rock...

... that is the question.









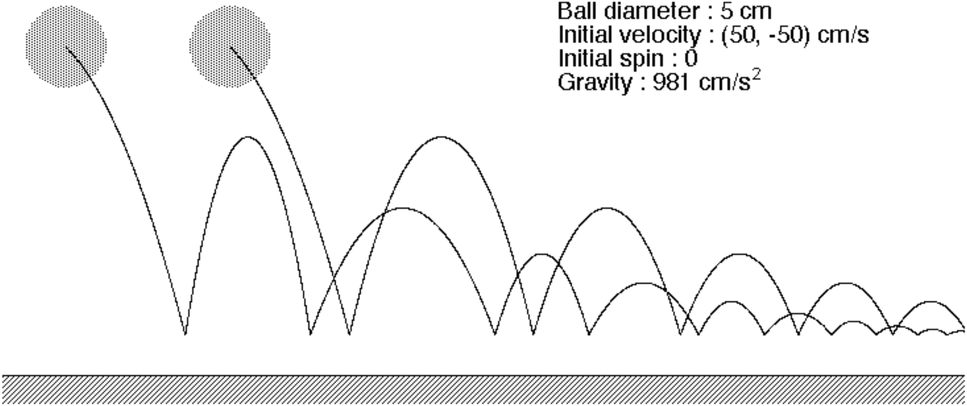
THE ROCKING BLOCK

While, in case of rounded edge...

... no velocity jump is expected

EFFECT OF TANGENTIAL RESTITUTION

Normal restitution coefficient : 0.8 for both balls.
Tangential restitution coefficient : 0.8 for ball on the left, 0 for ball on the right
Friction coefficient : 0.5



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